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SYNCHRONIZING CHAOS: EXPLORING ATTRACTIVE SETS IN A NOVEL 4D HYPERCHAOTIC LORENZ MODEL

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Abstract. Due to the very complex algebraic structure of hyperchaotic models, it is often difficult to determine their limits. Using Lyapunov's stability theory and optimization methods, we study the limits of a new 4D hyperchaotic Lorenz model. Based on the results obtained, we study complete chaotic synchronization. Finally, to demonstrate the effectiveness of the proposed chaotic synchronization scheme, some numerical simulations are provided.

Keywords: Chaos synchronization, Lagrange multiplier method, Lyapunov stability, boundedness of solutions, hyperchaotic system.

AMS Subject Classification: 65P20, 65P30, 65P40.

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1 Introduction

The research of hyperchaotic systems expanded into interdisciplinary areas in the years that followed, including physics, engineering, biology, and economics (Momani et al., 2023; Ouannas et al., 2022; Albadarneh et al., 2021; Batiha et al., 2022, 2021). Numerous hyperchaotic systems in nonlinear regimes have been discovered since the advent of the hyperchaotic Rosler system by Rosler, which suggests the existence of multiple positive Lyapunov exponents (Ouannas et al., 2021; Talbi et al., 2020; Bezziou et al., 2021). In these several fields, researchers have used hyperchaotic systems to simulate and comprehend complicated events. Because of their intrinsic complexity and pseudorandomness, hyperchaotic systems have also found use in secure communications, cryptography, and random number generation. The Lorenz-Haken system (Ning & Haken, 1990), the hyperchaotic Matsumoto circle (Matsumoto et al., 1986), the hyperchaotic modified Chua circle (Thamilmaran et al., 2004), the hyperchaotic Chua circle (Kapitaniak & Chua, 1994), and the hyperchaotic Chen system Chen et al. (2010) are typical examples of hyperchaotic systems. These examples demonstrate the wide range of applications of hyperchaos in fields like nonlinear circuits Cenys et al. (2003), synchronization (Jiang et al., 2004), secure

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communications (Udaltsov et al., 2003), neural networks (Arena et al., 1995), lasers (Vicente et al., 2005), control (Hsieh et al., 1999), and so on.

Compared to chaotic systems, hyperchaotic systems feature more unstable manifolds and are more complicated. In recent times, hyperchaos research has gained prominence in nonlinear science. One of the central problems of dynamical systems theory is identifying the boundaries between chaotic and hyperchaotic systems. It is still very difficult to predict whether hidden attractors exist in chaotic or hyperchaotic systems, which is particularly problematic for engineering applications where this information is crucial. Furthermore, measuring the fractal dimensions of chaotic attractors, like the Lyapunov and Hausdorff dimensions, and controlling chaos are all made possible by having a solid grasp of the boundedness of chaotic systems. Boundary estimate of chaotic and hyperchaotic systems has been studied recently using various approaches. But these techniques frequently target particular systems, which makes it difficult to develop a general strategy for estimating boundaries across different chaotic systems and prevents the limits of many systems from being found.

The use of different strategies and tactics to regulate or suppress chaotic activity in nonlinear dynamic systems is known as "chaos control". The goals of chaos management techniques are to stabilize these systems, forecast their behavior, or modify them for particular uses. A few popular methods for controlling chaos include feedback control, oscillating control, state-averaged control, optimum control, bifurcation control, and so on. Regarding control theory, see Imad & Abdelhamid (2016); Rezzoug & Ayadi (2018); Imad & Lamya (2022); Rezzoug & Ayadi (2017, 2023) for further information. For the purpose of researching the new chaotic system's qualitative behavior and chaos control, the system's boundary is crucial. A chaotic or hyperchaotic system cannot include a hidden attractor outside of the global set of attractions if we can demonstrate that the system contains a global set of attractions. This is critical for engineering applications (Leonov & Kuznetsov, 2013; Bragin et al., 2011) since it is highly uncertain whether hidden attractors will exist. Moreover, boundedness of chaotic systems is crucial for many applications such as chaos synchronization and control. The fractal dimension of chaotic attractors, like the Lyapunov and Hausdorff dimensions, can also be estimated as in Kuznetsov et al. (2014). Boundary estimate of chaotic and hyperchaotic systems has been the subject of numerous studies recently Elhadj & Sprott (2010); Leonov et al. (1987); Li et al. (2005, 2009); Pogromsky et al. (2003); Sun (2009); Wang et al. (2010); Fuchen (2019); Fuchen & Guangyun (2016); Gasimov et al. (2019). Nevertheless, the techniques chosen in the corresponding studies are limited to the corresponding systems. Deriving a universal approach to approximate the boundaries of any chaotic system is a highly challenging task. There are still many chaotic systems whose bounds are unknown.

The following references describes the complete synchronization of chaotic systems (A) and (B): Ouannas & Odibat (2015); Ouannas et al. (2017a, 2019, 2017b); Ouannas & Grassi (2016); Ouannas et al. (2017c, 2020). The goal of full synchronization is to control the slave system (B) so that its state asymptotically follows that of the main system (A). This is applicable if the chaotic system (A) is referred to as the master system or driving system, and the controlled chaotic system (B) is referred to as the slave system or response system. Because of their enormous potential applications in a variety of domains, including biology, engineering, medicine, and information technology, chaos control and synchronization have drawn a lot of attention. Numerous control systems, including B, have been created in recent decades to study global chaos synchronization difficulties. Methods such as sampled data feedback synchronization Yang & Chua (1999), OGY Ott et al. (1990), time-delay feedback Park & Kwon (2003), backstepping Yu & Zhang (2006), adaptive design Liao & Tsai (2000), sliding mode control Konishi et al. (1998), and so on.

Inspired by the conversation above, we will explore the boundaries of a novel hyperchaotic Lorenz system. After that, we analyze the full chaotic synchronization using the results that we have gathered. The system parameters are provided with an exact threshold by using a two-variable linear feedback controller. Lastly, we use numerical simulations to confirm the stability and effectiveness of the suggested chaotic synchronization approach. These simulations work as a testing ground, letting us observe the theoretical ideas in operation in a regulated computing setting. These simulations allow us to monitor the behavior of the system, evaluate the effects of changing parameters, and determine whether the synchronized states are stable. The numerical results provide a visual depiction of the synchronized trajectories and corroborate the analytical findings, making the theoretical framework more concrete and understandable for a broader audience. The core of this work is essentially the combination of theoretical investigation, rigorous analysis, and computational simulations, which opens up new avenues for our knowledge of hyperchaotic systems and the dynamics of synchronization.

2 Problem formulation and main result

The well-known Lorenz system is explained by Lorenz (1963):

$$\left\{ \begin{array}{c} x' = a\,(y-x) \\ y' = cx - xz - y \\ z' = xy - bz \end{array} \right. , \tag{1}$$

where x, y and z are the state variables, and a, b and c are the real constants. System (1) is chaotic when $a = 10$, $b = 8/3$, and $c = 28$. Three nonlinearity terms are included in the innovative hyperchaotic Lorenz system that Xingyuan and Mingjun Wang build in Wang & Wang (2008). The following system can be used to characterize the new four-dimensional system:

$$\left\{ \begin{array}{c} x' = a\,(y-x) + w \\ y' = cx - xz - y \\ z' = xy - bz \\ w' = rw - yz \end{array} \right. , \tag{2}$$

where a, b, c and r are all real constant parameters. The computation reveals that system (2) has the following Lyapunov exponents $\lambda_1 = 0.3381 > 0$, $\lambda_2 = 0.1586 > 0$, $\lambda_3 = 0$, and $\lambda_4 = -15.1752$, when $a = 10$, $b = \frac{8}{3}$, $c = 28$, and $r = -1$ are selected (see Wang & Wang (2008)). The hyperchaotic nature of system (2) is indicated by the two positive Lyapunov exponents. Fig. 1 displays the attractor's projections.

In Wang & Wang (2008), several fundamental dynamical characteristics of the new four-dimensional hyperchaotic system (2) were examined. However, there are still a lot of system (2) properties that are unknown. We shall talk about the boundedness of the new hyperchaotic system (2) in the following.

Lemma 1. *If Γ is a set defined by*

$$\Gamma = \left\{ (y,z)\,/\,\frac{y^2}{b^2} + \frac{(z-c)^2}{c^2} = 1,\ b>0,\ c>0 \right\} \tag{3}$$

and $G = y^2 + z^2$, $H = y^2 + (z-2c)^2$, $(y,z) \in \Gamma$. Then we have

$$\max_{(y,z)\in\Gamma} G = \max_{(xy,z)\in\Gamma} H = \left\{ \begin{array}{c} \frac{b^4}{b^2-c^2},\ b \geq \sqrt{2}c \\ 4c^2,\ b < \sqrt{2}c \end{array} \right. . \tag{4}$$

Proof. By using the Lagrange multiplier approach, it may be simply determined. ☐

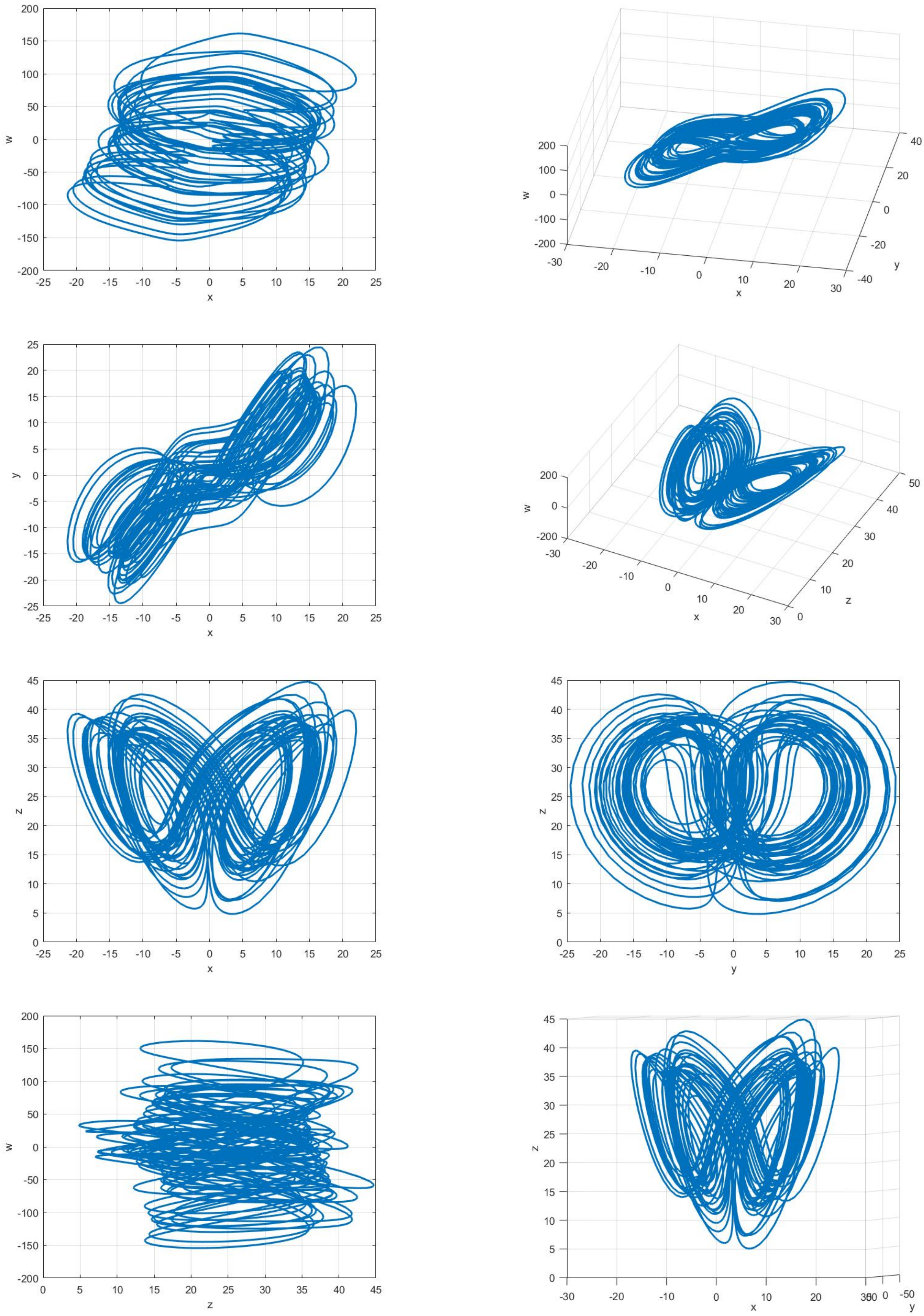

Figure 1: Hyperchaotic attractor of the system (2) with $a = 10$, $b = \dfrac{8}{3}$, $c = 28$ and $r = -1$

Theorem 1. *The following set*

$$\Omega = \left\{ (x, y, z, w) \, / , \; x^2 \leq \frac{\left(R^2 + Rc - raR\right)^2}{r^2 a^2} , y^2 + (z - c)^2 \leq R^2 , \; w^2 \leq \frac{\left(R^2 + Rc\right)^2}{r^2} \right\} \tag{5}$$

is the bound for systems (2), where

$$R^2 = \left\{ \begin{array}{l} \dfrac{b^2 c^2}{4(b-1)}, \quad if\, b \geq 2 \\ c^2 \;, \quad if\, b < 2 \end{array} \right. , \tag{6}$$

and where $a > 0$, $b > 0$, $c > 0$ and $r < 0$.

Proof. Define the following Lyapunov function

$$V_1 (y, z) = y^2 + (z - c)^2 . \tag{7}$$

Next, the time derivative along system (2)'s orbits is

$$\begin{aligned} \dot{V}_1 &= 2yy' + 2(z - c) z' \\ &= -2y^2 - 2bz^2 + 2cbz \\ &= -2y^2 - 2b\left(z - \frac{c}{2}\right)^2 + \frac{bc^2}{2} \end{aligned} \tag{8}$$

Now, consider the equation $\dot{V}_1 = 0$. This implies that the surface

$$\Gamma = \left\{ (y, z) \, / \frac{y^2}{\frac{bc^2}{4}} + \frac{\left(z - \frac{c}{2}\right)^2}{\frac{c^2}{4}} = 1, b > 0, c > 0 \right\} \tag{9}$$

will be an ellipsoid in 2D-space for certain values of b and c. It should be noted here that we have $\dot{V}_1 < 0$ outside Γ, while we have $\dot{V}_1 > 0$ inside Γ. Since the function $V_1 = y^2 + (z - c)^2$ is continuous on the losed set Γ, then V_1 can reach its maximum on the surface Γ. Now, denote the maximum value of V as R^2, that is

$$R^2 = \max V_{1(y,z)\in\Gamma}.$$

As a result, Lemma 1 makes it simple to get

$$V_1 (y, z) \leq R^2 = \left\{ \begin{array}{l} \dfrac{b^2 c^2}{4(b-1)}, \quad \text{if } b \geq 2 \\ c^2 \;, \quad \text{if } b < 2 \end{array} \right. . \tag{10}$$

From formula (10), we get

$$|y| \leq R, \; |z| \leq R + c. \tag{11}$$

Concurrently, system (2) and formula (11) yield

$$w' = rw - yz \leq rw + |y| \, |z| \leq rw + R(R + c) .$$

Using the principle of comparison, we arrive at

$$w(t) \leq \frac{R^2 + Rc}{-r} + \left(w(t_0) + \frac{R^2 + Rc}{r} \right) e^{r(t - t_0)}, \text{ where } r < 0. \tag{12}$$

So, we have

$$\lim_{t\to+\infty} w(t) \le \frac{R^2+Rc}{-r}. \tag{13}$$

In other words, the inequality $w^2 \le \frac{\left(R^2+Rc\right)^2}{r^2}$ is satisfied as $t \to +\infty$. Now, similarly, from (2), (11) and (13), we can obtain

$$x' = a(y-x) + w \le -ax + aR + \frac{R^2+Rc}{-r}. \tag{14}$$

Utilizing the comparison concept gives

$$x(t) \le \frac{-raR+R^2+Rc}{-ra} + \left(x(t_0) + \frac{-raR+R^2+Rc}{-ra}\right) e^{-a(t-t_0)}, \text{ where } r<0. \tag{15}$$

So, we have

$$\lim_{t\to+\infty} x(t) \le \frac{-raR+R^2+Rc}{-ra}. \tag{16}$$

In other words, the inequality $x^2 \le \frac{\left(R^2+Rc-raR\right)^2}{r^2a^2}$ holds as $t \to +\infty$. Therefore, we have the conclusion that

$$\Omega = \left\{ (x,y,z,w)\,/,\; x^2 \le \frac{\left(R^2+Rc-raR\right)^2}{r^2a^2}, y^2 + (z-c)^2 \le R^2,\; w^2 \le \frac{\left(R^2+Rc\right)^2}{r^2} \right\} \tag{17}$$

is the bound for the hyperchaotic systems (2), which finishes the proof of this result. □

3 Application in Chaos Synchronization

In this section, we will use the results obtained in the previous section to study chaos synchronization via linear feedback. For the master system (2), we construct another system called the slave system, which can be designed as

$$\begin{cases} \dot{x}_1 = ay_1 - ax_1 + w_1 - k_1(w_1 - w) \\ \dot{y}_1 = cx_1 - x_1z_1 - y_1 - k_2(y_1 - y) \\ \dot{z}_1 = x_1y_1 - bz_1 \\ \dot{w}_1 = -y_1z_1 + rw_1 \end{cases}, \tag{15}$$

where x_1, y_1, z_1, w_1 are the state variables and $k_1 > 0,\ k_2 > 0$ are the control parameters. From Theorem 2, we obtain

$$|y| \le R,\ \ |z| \le R + c. \tag{16}$$

Theorem 2. *Systems (2) and (15) are globally and asymptotically synchronized when*

$$k_2 > \frac{b(a\sigma + R + 2c)^2}{4ab\sigma - R^2} - 1,\ k_1 = 1,\ \left(\sigma > \frac{R^2}{4ab} > 0\right). \tag{17}$$

Proof. The complete synchronization error is described as follows:

$$e_1 = x_1 - x,\ e_2 = y_1 - y,\ e_3 = z_1 - z,\ e_4 = w_1 - w.$$

Then, the error dynamics is obtained as

$$\begin{cases} \dot{e}_1 = ae_2 - ae_1 \\ \dot{e}_2 = (c-z)e_1 - xe_3 - e_1e_3 - (k_2+1)e_2 \\ \dot{e}_3 = ye_1 + xe_2 + e_1e_2 - be_3 \\ \dot{e}_4 = re_4 - ye_3 - ze_2 - e_2e_3 \end{cases}. \tag{18}$$

Now, define the following Lyapunov function:

$$V(e_1, e_2, e_3) = \sigma e_1^2 + e_2^2 + e_3^2,$$

where σ is a positive constant and $\sigma > \dfrac{R^2}{4ab} > 0$. Therefore, along the system (18), its time derivative is provided by

$$\begin{aligned}
\frac{1}{2}\dot{V} &= \sigma e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 \\
&= \sigma e_1 (a e_2 - a e_1) + e_2 \left((c - z) e_1 - x e_3 - e_1 e_3 - (k_2 + 1) e_2\right) + e_3 (y e_1 + x e_2 + e_1 e_2 - b e_3) \\
&= -\sigma a e_1^2 - (k_2 + 1) e_2^2 - b e_3^2 + (\sigma a + c - z) e_1 e_2 + y e_1 e_3 \\
&\le -\sigma a e_1^2 - (k_2 + 1) e_2^2 - b e_3^2 + (a\sigma + R + 2c) |e_1| |e_2| + R |e_1| |e_3| \\
&= -E^T P E,
\end{aligned}$$

where

$$E = [|e_1|, |e_2|, |e_3|]^T, \ P = \begin{bmatrix} \sigma a & -\dfrac{a\sigma + R + 2c}{2} & -\dfrac{R}{2} \\ -\dfrac{a\sigma + R + 2c}{2} & k_2 + 1 & 0 \\ -\dfrac{R}{2} & 0 & b \end{bmatrix}$$

in which the matrix P is positive definite when

$$\sigma > \frac{R^2}{4ab} > 0, \text{ and } k_2 > \frac{b (a\sigma + R + 2c)^2}{4ab\sigma - R^2} - 1.$$

Thus, according to Lyapunov function theory, it follows that

$$\lim_{t \to +\infty} |e_1| = 0, \lim_{t \to +\infty} |e_2| = 0, \lim_{t \to +\infty} |e_3| = 0. \tag{19}$$

In the following content, we will prove that $\lim_{t \to +\infty} e_4 = 0$. To do so, we can find $\lim_{t \to +\infty} e_1 = 0$ based on (19). Therefore, there is a sufficiently large $T > t_0$ such that $\left|\dfrac{y e_3 + z e_2 + e_2 e_3}{-r}\right| < \varepsilon$ when $t \ge T$, for any $\varepsilon > 0$. So, when $t \ge T$ and from (18), we can have

$$\begin{aligned}
e_4(t) &= e_4(t_0) e^{r(t - t_0)} + e^{rt} \int_{t_0}^{t} (-y e_3 - z e_2 - e_2 e_3) e^{-r\tau} d\tau \\
&\le e_4(t_0) e^{r(t - t_0)} + e^{rt} \int_{t_0}^{t} (-r) \varepsilon e^{(-r)\tau} d\tau \\
&= (e_4(t_0) - \varepsilon) e^{r(t - t_0)} + \varepsilon,
\end{aligned}$$

for any $\varepsilon > 0$. Therefore, if the initial value $e_4(t_0) > \varepsilon$ and $t \to +\infty$, we obtain

$$e_4(t) - \varepsilon \le (e_4(t_0) - \varepsilon) e^{r(t - t_0)} \to 0.$$

Also, we have

$$\begin{aligned}
e_4(t) &= e_4(t_0) e^{r(t - t_0)} + e^{rt} \int_{t_0}^{t} (-y e_3 - z e_2 - e_2 e_3) e^{-r\tau} d\tau \\
&\ge e_4(t_0) e^{r(t - t_0)} - e^{rt} \int_{t_0}^{t} (-r) \varepsilon e^{(-r)\tau} d\tau \\
&= (e_4(t_0) + \varepsilon) e^{r(t - t_0)} - \varepsilon.
\end{aligned}$$

Therefore, if the initial value $e_4(t_0) < -\varepsilon$ and $t \to +\infty$, we get

$$e_4(t) + \varepsilon \leq (e_4(t_0) + \varepsilon)\, e^{r(t-t_0)} \to 0.$$

Consequently, when the initial value $|e_4(t_0)| > \varepsilon$ and $t \to +\infty$, we have the distance $d\,(e_4(t), I) \to 0$, where $I = [-\varepsilon, \varepsilon]$. So, for any sufficiently small $\varepsilon > 0$, there is a sufficiently large $T > t_0$ such that when $t > T$, we have $|e_4(t)| < \varepsilon$. By the de definition of limit, we obtain

$$\lim_{t\to+\infty} e_4(t) = 0. \tag{20}$$

Summarizing the above, we have

$$\lim_{t\to+\infty} |e_1| = 0, \lim_{t\to+\infty} |e_2| = 0, \lim_{t\to+\infty} |e_3| = 0, \lim_{t\to+\infty} |e_4| = 0.$$

Ultimately, we deduce that there is worldwide synchronization between the slave system (15) and the master system (2). The proof is now complete. □

It should be noted that k_1 and k_2 reported in (17) are not limited because they have been chosen as $k_1 > 0$ and $k_2 > 0$. Also, in matrix P reported in the above theorem, we observe that $P_22 = k_2 + 1$, which makes one to consider $\lim\limits_{t\to\infty} |e_2| = 0$. This implies $k_2 > \frac{b(a\sigma+R+2c)^2}{4ab\sigma-R^2} - 1$.

4 Simulation studies

In this part, using the MATLAB 7.4, some numerical simulations are presented. The initial conditions of master and slave systems are selected as $(-1, 1, -2, -3)$ and $(-7, 2, 3, 2)$, respectively. When $a = 10$, $b = 8/3$, $c = 28$, $r = -1$, it is easy to obtain $R^2 = \dfrac{b^2c^2}{4\,(b-1)} = \dfrac{112^2}{15}$, $\sigma > \dfrac{R^2}{4ab} = 7.84$, $k_2 > \dfrac{b\,(a\,(\sigma+1) + R + 2c)^2}{4ab\sigma - R^2} - 1 = 4248.3384$ according to the Theorem 2 and Theorem 3. Choose $\sigma = 8$, $k_2 = 4249$, then, the master system (2) synchronizes with the slave system (15) as shown in Figure 2 and Figure 3.

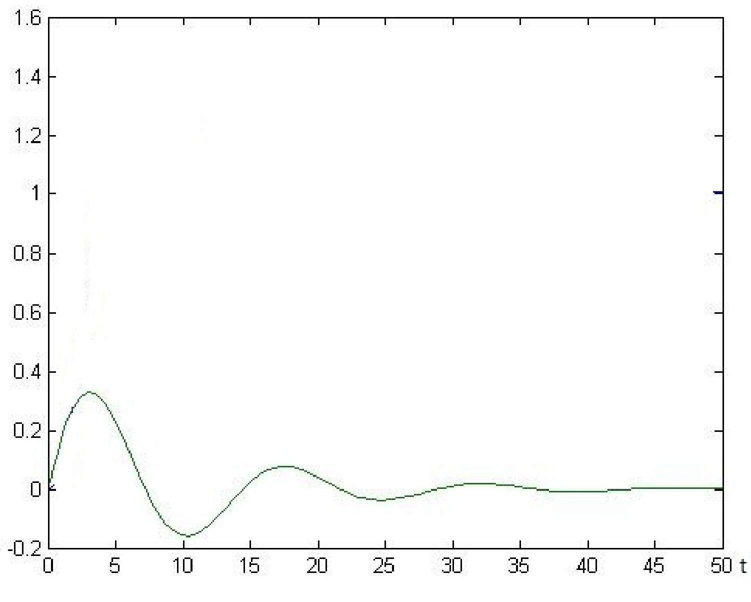

Figure 2: Effectiveness of chaos synchronization between system (2) and system (15).

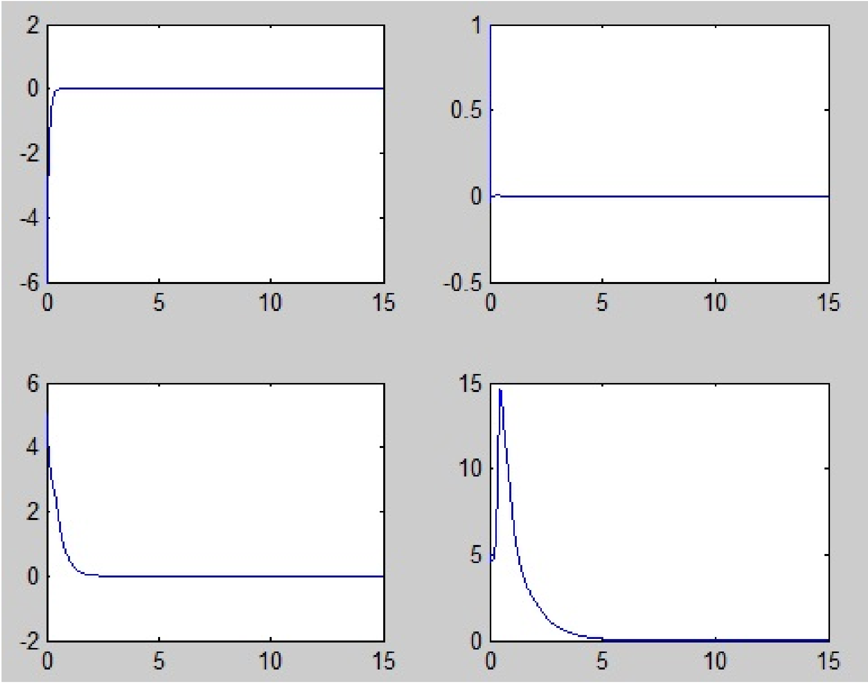

Figure 3: Effectiveness of chaos synchronization between system (2) and system (15).

5 Conclusion

In this research, understanding the limitations of chaotic systems is crucial for both control theory and practical applications. We employ Lyapunov function theory and an optimization method to establish boundaries for a newly identified 4D hyperchaotic system. These results enable us to achieve global chaos synchronization using a linear feedback approach with two inputs, a fundamental process for a wide range of applications. This synchronization transforms initial disorder into a harmonized behavior. By validating these theoretical boundaries, our study bridges the gap between theoretical concepts and real-world applications. To demonstrate the robustness of our framework, extensive numerical simulations are conducted, closely aligning with our theoretical analyses. The consistency observed between theory and simulations strengthens the validity of our findings, highlighting the effectiveness of our methods. This agreement between theory and simulations confirms the practical feasibility of our approach, offering a concrete pathway for implementing chaos control strategies in diverse technological and scientific fields.

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